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Convergence from Approximating Subspaces

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1. INTRODUCTION

Curtis [3] has considered the following problem. For each positive integer n, let E_n be a finite subset of the closed interval $[-1, 1]$ containing at least n points. For each real-valued continuous function x on $[-1, 1]$, let $P_n(x)$ denote the unique polynomial of degree at most $n-1$ which best approximates x uniformly over the set E_n. Letting $\|x\| = \sup\{|x(t)| \mid -1 \leqslant t \leqslant 1\}$, Curtis's main theorem states that the following two conditions are equivalent:

(1) $\|x - P_n(x)\| \to 0$ for each x continuous on $[-1, 1]$;

(2) There exists a constant K such that, for each $n = 1, 2,...$, if p is polynomial of degree at most $n-1$ and $|p(t)| \leqslant 1$ for all $t \in E_n$, then $\|p\| \leqslant K$.

A classical result of Faber [6] states that if each E_n contains exactly n points, then (1) fails for some x. Curtis [3; Theorem 1] shows that (1) also fails if each E_n contains at most $n+1$ points. On the other hand, a result of Bernstein [1; pp. 55–57] states that if $\lambda > 1$ is fixed and $m_n > \lambda n$ for every n, then a sequence (E_n) of subsets of $[-1, 1]$, with E_n containing m_n points, can be chosen so that (2) is satisfied.

It is the purpose of this note to present a generalization of Curtis's theorem to an arbitrary Banach space setting (Theorem 2.5). It is interesting to note that this theorem is a type of "uniform boundedness" principle, except that is applies to a certain sequence of (generally *nonlinear*) metric

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projections. One consequence of this result is the Erdös–Turán Theorem [5] which states that a certain sequence of interpolating polynomials to a given continuous function on [0, 1] converges, in the L_2-norm, to the function (Example 2.9). In Section 3, a variant of Theorem 2.5 is established (Theorem 3.1). This theorem is also related to one of Kripke [7] which states that to find a best approximation from a finite dimensional subspace of a normed linear space X to a given element in X, it is possible to replace this by the (often easier) problem of finding best approximations relative to a sequence of seminorms $\|\cdot\|_k$ on X with $\|\cdot\|_k \to \|\cdot\|$. Several examples are given to show that the hypotheses in Theorem 3.1 cannot be dropped.

2. A Convergence Theorem

In this section, unless otherwise stated, we assume the following hypotheses:

(i) X is a normed linear space;

(ii) (M_n) is an increasing sequence of finite dimensional subspaces of X;

(iii) (Γ_n) is a sequence of finite dimensional subspaces of the dual space X^*;

(iv) for each positive integer n, a seminorm on X is defined by

$$\|x\|_n = \sup\{|f(x)| \mid f \in \Gamma_n, \|f\| \leqslant 1\};$$

(v) each M_n is $\|\cdot\|_n$-Chebyshev, i.e., for each $x \in X$ there is a unique point $P_n(x) \in M_n$ such that

$$\|x - P_n(x)\|_n = d_n(x) \equiv \inf\{\|x - y\|_n \mid y \in M_n\}.$$

The mapping $x \mapsto P_n(x)$ is called the metric projection onto M_n relative to the seminorm $\|\cdot\|_n$. It is easy to verify that P_n is homogeneous, additive modulo M_n, and idempotent (i.e., $P_n(\alpha x) = \alpha P_n(x)$, $P_n(x + y) = P_n(x) + y$ for all $x \in X$, $y \in M_n$, and $P_n^2 = P_n$), but P_n is not linear in general. The *norm* of P_n is defined by

$$\|P_n\| = \sup\{\|P_n(x)\| \mid x \in X, \|x\| \leqslant 1\}.$$

By the homogeneity of P_n, it follows that $\|P_n(x)\| \leqslant \|P_n\| \, \|x\|$ for every x.

2.1. Lemma. *The seminorm $\|\cdot\|_n$ is actually a norm on M_n. That is, $y \in M_n$ and $\|y\|_n = 0$ implies $y = 0$.*

Proof. Suppose $\|y\|_n = 0$ for some $y \in M_n$. Then

$$\|x - (P_n(x) + y)\|_n \leqslant \|x - P_n(x)\|_n + \|y\|_n = \|x - P_n(x)\|_n.$$

By uniqueness of best approximations, $P_n(x) + y = P_n(x)$, i.e., $y = 0$. ∎

The next lemma is another way of stating that each mapping P_n is an "open mapping" with the same "openness constant" (viz. 2).

2.2. LEMMA. *For each positive integer n and each $y \in M_n$ with $\|y\|_n \leqslant 1$, there exists an $x \in X$ with $\|x\| \leqslant 2$ and $P_n(x) = y$.*

Proof. Given $y \in M_n$ with $\|y\|_n \leqslant 1$, define $G = G_y$ on Γ_n by

$$G(f) = f(y) \qquad (f \in \Gamma_n).$$

Then G is linear and

$$\sup_{\substack{f \in \Gamma_n \\ \|f\| = 1}} |G(f)| = \sup_{\substack{f \in \Gamma_n \\ \|f\| = 1}} |f(y)| = \|y\|_n \leqslant 1.$$

Thus $G \in \Gamma_n^*$ and $\|G\| \leqslant 1$. By the Hahn–Banach theorem G has a norm-preserving extension (also denoted by G) in X^{**}. Since Γ_n is finite dimensional, Helly's theorem (see, e.g., [4; pp. 86, 87]) implies that there is an $x \in X$ with $f(x) = G(f)$ for $f \in \Gamma_n$, and $\|x\| \leqslant \|G\| + 1 \leqslant 2$. Hence

$$f(x - y) = f(x) - f(y) = G(f) - G(f) = 0$$

for all $f \in \Gamma_n$. Thus $\|x - y\|_n = 0$ and hence $y = P_n(x)$. ∎

From Lemma 2.1 and the fact that all norms on a finite dimensional space are equivalent, it follows that there is a constant K_n such that

$$\|y\| \leqslant K_n \|y\|_n \qquad (y \in M_n).$$

The next result gives a condition equivalent to when a *single* constant works for every n.

2.3. LEMMA. *The following statements are equivalent.*

(1) *There is a constant K such that, for every n, $\|y\| \leqslant K\|y\|_n$ $(y \in M_n)$;*

(2) *There is a constant K such that, for every n, $y \in M_n$ and $\|y\|_n \leqslant 1$ implies $\|y\| \leqslant K$;*

(3) $\sup_n \|P_n\| < \infty$.

Proof. The equivalence of (1) and (2) is obvious.

$(1) \Rightarrow (3)$. Assuming condition (1), we have

$$\|P_n\| = \sup_{\|x\| \leqslant 1} \|P_n(x)\| \leqslant \sup_{\|x\| \leqslant 1} K \|P_n(x)\|_n = K \sup_{\|x\| \leqslant 1} \|P_n(x)\|_n.$$

If $\|x\| \leqslant 1$, then

$$\|P_n(x)\|_n \leqslant \|P_n(x) - x\|_n + \|x\|_n \leqslant 2\|x\|_n \leqslant 2\|x\| \leqslant 2.$$

So $\|P_n\| \leqslant 2K$. Thus (3) holds.

$(3) \Rightarrow (2)$. Let $K = 2\sup_n \|P_n\|$, $y \in M_n$, and $\|y\|_n \leqslant 1$. By Lemma 2.2, there exists $x \in X$ with $P_n(x) = y$ and $\|x\| \leqslant 2$. Thus

$$\|y\| = \|P_n(x)\| \leqslant \|P_n\| \|x\| \leqslant K. \quad \blacksquare$$

2.4. LEMMA. *Consider the following statements.*

(1) $\sup_n \|P_n\| < \infty$;

(2) $\sup_n \|P_n(x)\| < \infty$ *for every* $x \in X$;

(3) $\lim_n \|x - P_n(x)\| = 0$ *for every* $x \in X$.

Then $(1) \Rightarrow (2)$ *and* $(3) \Rightarrow (2)$.

Suppose, moreover, that $\bigcup_1^\infty M_n$ *is dense in* X. *Then* $(1) \Rightarrow (3)$ *and if, in addition,* X *is complete,* $(2) \Rightarrow (1)$. *In particular, if* $\bigcup_1^\infty M_n$ *is dense and* X *is complete, then all three statements are equivalent.*

Proof. The implications $(1) \Rightarrow (2)$ and $(3) \Rightarrow (2)$ are trivial.

For the remainder of the proof, we assume that $\bigcup_1^\infty M_n$ is dense in X.

$(1) \Rightarrow (3)$. Let $K = \sup_n \|P_n\|$, $x \in X$, and $\varepsilon > 0$. Choose $y \in \bigcup_1^\infty M_n$ so that $\|x - y\| < \varepsilon(1+K)^{-1}$. Then $y \in M_n$ for n sufficiently large so, for such n, using the additivity modulo M_n of P_n,

$$\begin{aligned} \|x - P_n(x)\| &\leqslant \|x - y\| + \|y - P_n(x)\| \\ &= \|x - y\| + \|P_n(y - x)\| \\ &< \varepsilon(1+K)^{-1} + K\varepsilon(1+K)^{-1} = \varepsilon. \end{aligned}$$

That is, (3) holds.

Now assume also that X is complete.

$(2) \Rightarrow (1)$. If (2) holds, define

$$X_k = \{x \in X \mid \sup_n \|P_n(x)\| \leqslant k\}.$$

Clearly, $X = \bigcup_1^\infty X_k$. By the standard compactness argument that shows that the (usual) metric projection onto a finite dimensional Chebyshev subspace is continuous, one can verify that P_n is $\|\cdot\|$ to $\|\cdot\|_n$ continuous, and hence (using Lemma 2.1 and the equivalence of norms on M_n), P_n is $\|\cdot\|$ to $\|\cdot\|$ continuous. From this fact it follows that X_k is closed. By the Baire Category Theorem, there is an integer k_0, an $x_0 \in X_{k_0}$, and $\varepsilon > 0$ so that the ball

$$B(x_0, \varepsilon) \equiv \{x \in X \mid \|x - x_0\| < \varepsilon\}$$

is contained in X_{k_0}. By the denseness of $\bigcup_1^\infty M_n$, we may assume that $x_0 \in M_N$ for some N. We have

$$\sup_n \|P_n(y)\| \leqslant k_0 \qquad (y \in B(x_0, \varepsilon)).$$

Thus if $n \geqslant N$ and $y \in B(x_0, \varepsilon)$, then $x_0 \in M_n \cap X_{k_0}$ so $x_0 = P_n(x_0)$ and

$$\|P_n(y - x_0)\| = \|P_n(y) - x_0\| \leqslant \|P_n(y)\| + \|x_0\| \leqslant 2k_0.$$

Hence for $n \geqslant N$ and $z \in X$ with $\|z\| < \varepsilon$, $y = z + x_0 \in B(x_0, \varepsilon)$ so

$$\|P_n(z)\| = \|P_n(y - x_0)\| \leqslant 2k_0.$$

It follows by homogeneity of P_n that

$$\|P_n(u)\| \leqslant \frac{2k_0}{\varepsilon} \qquad \text{for all} \quad u \in B(0, 1).$$

Thus $\|P_n\| \leqslant 2k_0/\varepsilon$ for $n \geqslant N$ implies $\sup_n \|P_n\| < \infty$. ∎

Remark. Note that the equivalence of (1) and (2) is a "uniform boundedness" principle for the (generally nonlinear) operators P_n.

Combining Lemmas 2.3 and 2.4 we immediately obtain the main result.

2.5. THEOREM. *Let X be a Banach space and suppose $\bigcup_1^\infty M_n$ is dense in X. Then the following statements are equivalent.*

(1) *There exists a constant K such that, for each n, $y \in M_n$ and $\|y\|_n \leqslant 1$ imply $\|y\| \leqslant K$;*

(2) *There exists a constant K such that, for each n and each $y \in M_n$, $\|y\| \leqslant K\|y\|_n$;*

(3) $\sup_n \|P_n(x)\| < \infty$ *for every* $x \in X$;

(4) $\sup_n \|P_n\| < \infty$;

(5) $\lim_n \|x - P_n(x)\| = 0$ *for every* $x \in X$.

For the following result, let T be a locally compact Hausdorff space and let $C_0(T)$ denote the linear space of all real-valued continuous functions x on

T "vanishing at infinity," i.e., $\{t \in T \mid |x(t)| \geqslant \varepsilon\}$ is compact for each $\varepsilon > 0$. With the norm $\|x\| = \sup\{|x(t)| \mid t \in T\}$, $C_0(T)$ is a Banach space. If T is actually compact, then $C_0(T)$ reduces to the space of all real-valued continuous functions on T, and is also denoted by $C(T)$. Let $(M_n)_{n=1}^{\infty}$ be an increasing sequence of finite dimensional Haar subspaces of $C_0(T)$ whose union $\bigcup_1^\infty M_n$ is dense in $C_0(T)$. (Recall that an n dimensional subspace M of $C_0(T)$ is called a Haar subspace iff each nonzero element of M has at most $n-1$ zeros. Furthermore, a finite dimensional subspace of $C_0(T)$ is a Haar subspace iff it is a Chebyshev subspace.) For each integer n, let E_n be a finite subset of T which contains at least $\dim M_n$ points. For each n we define a seminorm on $C_0(T)$ by

$$\|x\|_n = \sup\{|x(t)| \mid t \in E_n\}.$$

For a given $x \in C_0(T)$, let $P_n(x)$ denote the *unique* element of M_n which is closest to x relative to the seminorm $\|\cdot\|_n$:

$$\|x - P_n(x)\|_n = \inf\{\|x - y\|_n \mid y \in M_n\}.$$

(This makes sense since $M_n|_{E_n}$ is a Haar subspace in $C_0(T)|_{E_n} = C(E_n)$.)

2.6. COROLLARY. *The following statements are equivalent.*

(1) *There is a constant K such that, for each n, $y \in M_n$ and $|y(t)| \leqslant 1$ for all $t \in E_n$ implies $\|y\| \leqslant K$;*

(2) $\sup_n \|P_n\| < \infty$;

(3) $\sup_n \|P_n(x)\| < \infty$ *for each* $x \in C_0(T)$;

(4) $\lim_n \|x - P_n(x)\| = 0$ *for each* $x \in C_0(T)$.

Proof. We will exhibit a sequence of finite dimensional subspaces Γ_n of the dual space $C_0(T)^*$ such that for each n and each $x \in C_0(T)$,

$$\sup\{|x(t)| \mid t \in E_n\} = \sup\{|f(x)| \mid f \in \Gamma_n, \|f\| \leqslant 1\},$$

i.e., $\|x\|_n = \sup\{|f(x)| \mid f \in \Gamma_n, \|f\| \leqslant 1\}$. Having done this, the result is then an immediate consequence of Theorem 2.5. Let

$$\Gamma_n = \operatorname{span}\{\delta_t \mid t \in E_n\},$$

where δ_t denotes the functional "evaluation at t." For each $x \in C_0(T)$, one has

$$\left| \sum_{t_i \in E_n} \alpha_i \delta_{t_i}(x) \right| \leqslant \sum |\alpha_i|\, |x(t_i)| \leqslant \sum |\alpha_i|\, \|x\|_n \leqslant \sum |\alpha_i|\, \|x\|. \quad (2.6.1)$$

On the other hand, by Urysohn's lemma we can choose $x \in C_0(T)$ with $\|x\| \leqslant 1$ and $x(t_i) = \operatorname{sgn} \alpha_i$ for all $t_i \in E_n$. Thus

$$\left| \sum_{t_i \in E_n} \alpha_i x(t_i) \right| = \sum |\alpha_i|. \tag{2.6.2}$$

Using relations (2.6.1) and (2.6.2), we get that $\|\sum_{t_i \in E_n} \alpha_i \delta_{t_i}\| = \sum |\alpha_i|$, and

$$\begin{aligned}
&\sup\{|f(x)| \mid f \in \Gamma_n, \|f\| \leqslant 1\} \\
&= \sup \left\{ \left| \sum \alpha_i x(t_i) \right| \;\middle|\; t_i \in E_n, \sum |\alpha_i| \leqslant 1 \right\} \\
&\leqslant \sup\{|x(t_i)| \mid t_i \in E_n\} \\
&\leqslant \sup \left\{ \left| \sum \alpha_i x(t_i) \right| \;\middle|\; t_i \in E_n, \sum |\alpha_i| \leqslant 1 \right\} \\
&= \sup\{|f(x)| \mid f \in \Gamma_n, \|f\| \leqslant 1\}.
\end{aligned}$$

Thus

$$\sup\{|f(x)| \mid f \in \Gamma_n, \|f\| \leqslant 1\} = \sup\{|x(t)| \mid t \in E_n\}. \quad \blacksquare$$

2.7. *Remarks.* (1) If M_n is an n dimensional Haar subspace in $C[a, b]$ and E_n is a subset of $[a, b]$ consisting of n points (resp. $n+1$ points), then $P_n(x)$ is the unique element in M_n interpolating x on E_n (resp. M_n is a hyperplane in $C(E_n)$). In either case, P_n is linear. By a result of Kharshiladze and Lozinski (cf., e.g., [2; p. 214]) condition (2) of Corollary 2.6 fails. Thus condition (4) also fails. This last remark yields an alternate proof to a result of Curtis [3; Theorem 1] (who stated it in the particular case when $T = [-1, 1]$ and $M_n = \prod_{n-1}$ is the space of polynomials of degree at most $n-1$).

(2) In the particular case when $T = [-1, 1]$ and $M_n = \prod_{n-1}$, Curtis proved the equivalence of conditions (1) and (4) in Corollary 2.6 [3; Theorem 2].

We next give two "indirect" applications of Theorem 2.5. These applications are indirect because the seminorms are not defined by finite dimensional subspaces Γ_n of the dual space; however, since the validity of Lemma 2.2 and the inequality $\|x\|_n \leqslant \|x\|$ can be readily verified, Lemma 2.3 and hence Theorem 2.5 are applicable.

2.8. EXAMPLE. Let $X = C[0, 1]$ and $M_n = \prod_{n-1}$ $(n = 2, 3, \ldots)$. For every integer $n \geqslant 2$, let $p = p(n)$ be the smallest even integer such that for every $y \in M_n \backslash \{0\}$,

$$\|y\|_p / \|y\| > 1 - 1/n, \tag{2.8.1}$$

where $\|y\|_p = [\int_0^1 |y(t)|^p\,dt]^{1/p}$. Set $m = m(n) = \frac{1}{2}[(n-1)p+2]$. For every $x \in X$, define

$$\|x\|_n = \left[\sum_{i=1}^{m} a_{im} x^p(t_{im})\right]^{1/p}, \tag{2.8.2}$$

where $\{t_{1m}, t_{2m},..., t_{mm}\}$ are the roots of the mth orthogonal polynomial on $[0, 1]$ and the a_{im} $(i = 1, 2,..., m)$ are the Gaussian integration coefficients. For any $y \in M_n$, we have

$$\|y\|_p = \|y\|_n \tag{2.8.3}$$

since (2.8.2) is an exact integration formula for all polynomials of degree $2m-1$ $(>(n-1)p)$. Since $a_{im} > 0$ and $\sum_{i=1}^m a_{im} = 1$, it follows that

$$\|x\|_n \leqslant \|x\|, \qquad x \in X. \tag{2.8.4}$$

Given any $y \in M_n$ with $\|y\|_n \leqslant 1$, it follows from (2.8.1) and (2.8.3) that $\|y\| \leqslant 2\|y\|_n \leqslant 2$. Thus Lemma 2.2 holds with $x = y$. Using this and (2.8.4), it follows that Lemma 2.3 is valid. As mentioned in the preceding paragraph, Theorem 2.5 is now applicable. Thus we conclude: if $P_n(x)$ denotes the best approximation to x from M_n (relative to the seminorm $\|\cdot\|_n$), then

$$\lim_{n\to\infty} \|x - P_n(x)\| = 0, \qquad x \in X.$$

2.9. Example. Fix any even integer p. Let X denote the set of all real-valued continuous functions x on $[0, 1]$ with the norm $\|x\|_p$, where $\|x\|_p$, M_n, and $\|x\|_n$ are defined as in Example 2.8. (Note that X is *not* complete.) However, by an argument similar to that in 2.8 (where here (2.8.1) is replaced by $\|y\|_n/\|y\|_p = 1$, $y \in M_n$) we have that Lemmas 2.2, 2.3, and 2.4 are valid. Since $\bigcup_{n=2}^{\infty} M_n$ is dense in X, Lemmas 2.3 and 2.4 implies that $\lim_n \|x - P_n(x)\| = 0$ for all $x \in X$, i.e.,

$$\lim_{n\to\infty} \int_0^1 |x(t) - (P_n x)(t)|^p\,dt = 0. \tag{2.9.1}$$

In the particular case when $p = 2$, it follows that $m = n$, $P_n(x)$ is the polynomial of degree $n-1$ which interpolates to x at the points $t_{1n}, t_{2n},..., t_{nn}$, and the Erdös–Turán Theorem [5] results (see also [2; p. 137]).

3. A Variant of Theorem 2.5

In this section we will consider the case when $\bigcup_{n=1}^{\infty} M_n$ is not dense in X, i.e., $M \equiv \overline{\bigcup_n M_n}$ is a *proper* closed subspace of X. We will prove a variant of

Theorem 2.5. Then, by means of examples, we will show that each of the hypotheses is essential.

3.1. THEOREM. *Let X be a Banach space and let M_n, $\|\cdot\|_n$, and P_n be defined as in Section 2. Suppose that, for every $x \in X$, there is a subsequence $\{n_k\}$ of the natural numbers with*

(i) $\lim_{k\to\infty} \|x\|_{n_k} = \|x\|$, $x \in X$;

(ii) *there exists* $x_0 \in M \equiv \overline{\bigcup_{n=1}^{\infty} M_n}$ *such that* $\lim_{k\to\infty} \|x_0 - P_{n_k}(x)\|_{n_k} = 0$.

Then $\|x - x_0\| = d(x, M)$.
Suppose, in addition,

(iii) *one of the statements of Lemma* 2.3 *holds.*

Then $\lim_{k\to\infty} \|x_0 - P_{n_k}(x)\| = 0$ *and* $\lim_{k\to\infty} \|x - P_{n_k}(x)\| = d(x, M)$.

Proof. Let $y \in M$. Then

$$\|x - y\| \geqslant \|x - y\|_{n_k} \geqslant \|x - P_{n_k}(x)\|_{n_k} \geqslant \|x - x_0\|_{n_k} - \|x_0 - P_{n_k}(x)\|_{n_k}.$$

Passing to the limit as $k \to \infty$ and using (i) and (ii), we get $\|x - y\| \geqslant \|x - x_0\|$. Thus x_0 is a best approximation to x from M, i.e., $\|x - x_0\| = d(x, M)$.

Assume, in addition, (iii). Hence there exists a constant K such that, for every n and every $y \in M_n$, $\|y\| \leqslant K \|y\|_n$. Hence

$$\begin{aligned}\|x_0 - P_{n_k}(x)\| &\leqslant \|x_0 - P_{n_k}(x_0)\| + \|P_{n_k}(x_0) - P_{n_k}(x)\| \\ &\leqslant \|x_0 - P_{n_k}(x_0)\| + K \|P_{n_k}(x_0) - P_{n_k}(x)\|_{n_k} \\ &\leqslant \|x_0 - P_{n_k}(x_0)\| + K[\|P_{n_k}(x_0) - x_0\|_{n_k} + \|x_0 - P_{n_k}(x)\|_{n_k}] \\ &\leqslant (1 + K) \|x_0 - P_{n_k}(x_0)\| + K \|x_0 - P_{n_k}(x)\|_{n_k}.\end{aligned}$$

By (ii), $\|x_0 - P_{n_k}(x)\|_{n_k} \to 0$. By Lemma 2.3, $\sup_n \|P_n\| < \infty$ and hence, applying Lemma 2.4 to M instead of X, we deduce that $\|x_0 - P_{n_k}(x_0)\| \to 0$. Thus $\|x_0 - P_{n_k}(x)\| \to 0$. Finally,

$$\begin{aligned} d(x, M) \leqslant \|x - P_{n_k}(x)\| &\leqslant \|x - x_0\| + \|x_0 - P_{n_k}(x)\| \\ &= d(x, M) + \|x_0 - P_{n_k}(x)\| \to d(x, M)\end{aligned}$$

implies $\|x - P_{n_k}(x)\| \to d(x, M)$. ∎

3.2. COROLLARY. *Suppose that conditions* (i), (ii), *and* (iii) *of Theorem* 3.1 *hold. Then, for each* $x \in X$, *some subsequence of the sequence* $\{P_n(x)\}$ *converges (in the norm of* X*) to a best approximation to* x *from* M.

Kripke [7] has shown that if X is a *finite* dimensional normed linear space, M_0 is a subspace of X, $M_n = M_0$ $(n = 1, 2,...)$, and $\|\cdot\|_n$ is a seminorm on M_n, then condition (i) alone implies the conclusions of Theorem 3.1.

In contrast to this, it is shown in the following examples that *none* of the conditions (i), (ii), or (iii) can be dispensed with in general.

3.3. EXAMPLE. Fix a positive integer N, let $X = \prod_N$ (≡the polynomials of degree at most N), and $M_n = \prod_{N-1}$ $(n = 1, 2,...)$. Thus $M = \overline{\bigcup M_n} = \prod_{N-1}$. Let E be a finite subset of $[0, 1]$ such that, for every $x \in X$,

$$\|x\| = \sup\{|x(t)| \mid t \in [0, 1]\} \leqslant 2\,|||x|||,$$

where $|||x||| = \sup\{|x(t)| \mid t \in E\}$. Set $\|x\|_n = |||x|||$ for every n. Given $x \in X$ with $0 \leqslant x \leqslant 1$, $\|x\| = 1$, and $x|_E = 0$, it follows that $P_n(x) = 0$ for all n. Let x_0 be the best approximation to x from M over E:

$$\|x - x_0\|_n = \inf_{y \in M} \|x - y\|_n.$$

Thus $x_0 = 0$ and $\|x_0 - P_n(x)\| = 0 = \|x_0 - P_n(x)\|_n$ for all n. But

$$\|x - x_0\| = 1 > \tfrac{1}{2} \geqslant d(x, M)$$

since the constant function $y = \frac{1}{2}$ in M_n satisfies $\|x - y\| = \frac{1}{2}$. Thus the conclusion of Theorem 3.1 fails although conditions (ii) and (iii) hold.

3.4. EXAMPLE. Let

$$X = \{x + \alpha h \mid x \in C[-1, 1], -\infty < \alpha < \infty\},$$

where $h(t) = 1$ if $0 \leqslant t \leqslant 1$ and $h(t) = 0$ if $-1 \leqslant t < 0$. Endow X with the supremum norm. Let $E = \{t_i \mid i = 1, 2,...\}$ be a dense sequence in $[-1, 1]$ with $t_1 = -1$, $t_2 = 0$, and $t_3 = 1$. For each $n \geqslant 3$, define

$$E_n = \{t_i \mid i = 1, 2,..., n\} = \{t_i^{(n)} \mid i = 1, 2,..., n\},$$

where the $t_i^{(n)}$ are ordered: $t_1^{(n)} < t_2^{(n)} < \cdots < t_n^{(n)}$. Let M_n be the n dimensional subspace of X consisting of those functions which are linear in each of the intervals $[t_i^{(n)}, t_{i+1}^{(n)}]$ $(i = 1, 2,..., n-1)$ and continuous on $[-1, 1]$. Clearly, $\bigcup_3^\infty M_n$ is not dense in X. Define the seminorm $\|x\|_n = \sup\{|x(t)| \mid t \in E_n\}$. For any $x \in X$, the piecewise linear function $y \in M_n$ which agrees with x on E_n satisfies $\|x - y\|_n = 0$. Thus $P_n(x)(t) = x(t)$ for all $t \in E_n$. For the function h, $\|h - P_n(h)\| = 1$ for every n while $d(h, M) = \frac{1}{2}$. Hence the conclusion of Theorem 3.1 fails although conditions (i) and (iii) hold.

3.5. EXAMPLE. Let $X = C[-1, 1]$, $M_n = \prod_{n-1}$ $(n = 1, 2,...)$, and let $E = \{t_i \mid i = 1, 2,...\}$ be a dense sequence in $[-1, 1]$. Define $E_n = \{t_i \mid i = 1, 2,..., n\}$ and $\|x\|_n = \sup\{|x(t)| \mid t \in E_n\}$ $(n = 1, 2,...)$. Clearly, M_n is $\|\cdot\|_n$-Chebyshev. In fact, $P_n(x) \in M_n$ interpolates to x on E_n. Since $M = \overline{\bigcup_1^\infty M_n} = C[0, 1]$, we have $x_0 = x$ for every $x \in X$. By the result of Faber mentioned in the Introduction, the conclusion of Theorem 3.1 fails for some x. However, conditions (i) and (ii) hold.

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